CIF (Cauchy Integral Formula) f holom. $\alpha \to$ CCW around z

$$f(z) = \frac{1}{2\pi i}\int_\alpha \frac{f(w)}{w-z}\,dw$$

What properties can we get from this.

(2) Liouville's Theorem Let f be an entire (domain $= \mathbb{C}$.) holomorphic function. If f is bounded (i.e. $\exists M > 0$ s.t. $|f(z)| \le M \ \forall z \in \mathbb{C}$), then f is a constant function.

Proof: Sp. f is holom. on $\mathbb{C}$, $|f(z)| \le M \ \forall z \in \mathbb{C}$.

Then for any $z \in \mathbb{C}$, we have (by CIFD) ☆ see below ☆

$$f'(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{(w-z)^2}\,dw \quad , \text{ where}$$

$$C_R = \{w \in \mathbb{C} : |z - w| = R\} \text{ oriented CCW.}$$

where $R > 0$.

$$\text{Then } |f'(z_0)| = \frac{1}{2\pi}\left|\int_{C_R}\frac{f(w)}{(w-z)^2}\,dw\right| \qquad \text{(ML inequality)}$$

$$\le \frac{1}{\cancel{2\pi}}\left(\max_{C_R}\left|\frac{f(w)}{(w-z)^2}\right|\right) \cdot \cancel{2\pi} R$$

$$\Rightarrow \quad |f'(z)| \leq \frac{\max_{C_R}|f(w)|}{R^2}\cdot R \leq \frac{MR}{R^2} = \frac{M}{R}.$$

True for every z, and every $R>0$.

Let's fix z, and let $R\to\infty$. By the order limit thm.

$$\lim_{R\to\infty}|f'(z)| \leq \lim_{R\to\infty}\frac{M}{R} = 0.$$

$$\Rightarrow \quad |f'(z)| \leq 0 \Rightarrow f'(z)=0.$$

Since this is true $\forall z\in\mathbb{C}$, $f(z)$ is a constant. $\boxtimes$

Note: If we remove "entire", this is false. You can have bounded holomorphic fcns on a smaller domain.

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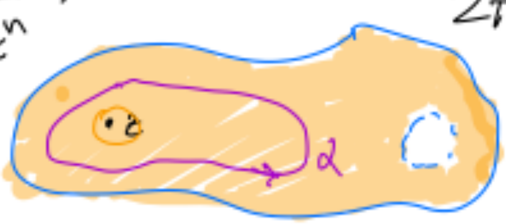
① CIFD (Cauchy Integral Formula for Derivatives)

If f is holom. in domain D, $z\in D$, α is CCW curve around z s.t. α is homotopic to a small circle around z in $D\setminus\{z\}$, then for all $n\geq 0$,

$$\frac{d^n f(z)}{dz^n} = f^{(n)}(z) = \frac{n!}{2\pi i}\int_\alpha \frac{f(w)}{(w-z)^{n+1}}\,dw.$$



Proof: with notation as above, we use CIF:

$$f(z) = \frac{1}{2\pi i}\int_\alpha \frac{f(w)}{(w-z)}\,dw.$$

Take $\frac{d}{dz}$ of both sides

[$\left(\frac{d}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right)$ of both sides]

$$\Rightarrow f'(z) = \frac{1}{2\pi i}\,\frac{d}{dz}\int_\alpha \frac{f(w)}{w-z}\,dw$$

technical result from analysis

$$= \frac{1}{2\pi i}\int_\alpha \frac{d}{dz}\left(\underbrace{\frac{f(w)}{w-z}}\right)dw$$

smooth in z, all its derivatives are bounded in abs value & absolutely integrable over α.

[technical fact: You can move a derivative of a parameter inside an integral if the abs value of that derivative of the integrand is integrable.]

$$\Rightarrow \quad f'(z) = \frac{1}{2\pi i}\int_\alpha \frac{f(\omega)}{(\omega-z)^2}\,d\omega$$

Now take n derivatives instead of first 1.

Side calculation $\frac{d^n}{dz^n}\left[(\omega-z)^{-1}\right] = \left[(\omega-z)^{-2}\right]^{(n-1)}$

$$= \left[(+2)(\omega-z)^{-3}\right]^{(n-2)}$$

$$\vdots$$

$$= (n!)\,(\omega-z)^{-n-1}.$$

can use induction to show.

$\Rightarrow$ CIF $\Rightarrow$

$$f^{(n)}(z) = \frac{1}{2\pi i}\int_\alpha (n!)(\omega-z)^{-n-1} f(\omega)\,d\omega$$

$$= \boxed{\frac{n!}{2\pi i}\int_\alpha \frac{f(\omega)}{(\omega-z)^{n+1}}\,d\omega}. \quad \boxed{//}$$

Example of the Consequences

- Fundamental Theorem of Algebra:

Every noncostant polynomial with complex # coefficients has a root in $\mathbb{C}$.

[Consequence: if $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ with $a_n \neq 0$, then $p(z) = a_n (z - z_1)(z - z_2) \cdots (z - z_n)$, where each $z_j \in \mathbb{C}$. Reason: If $p(z)$ has one root z_1, then you can find $q(z) = \frac{p(z)}{z - z_1}$ = $(n-1)$ degree polynomial by long division. Keep going to get the complete factorization.]

Proof (Sketch: full proof will be in the hmk) —
Assume $p(z)$ is an n^{th} degree polynomial ($n \geq 1$) that has no root. Consider $g(z) = \frac{1}{p(z)}$ ← entire function.
If we can show $|g(z)| \leq$ constant $\forall z \in \mathbb{C}$, then that will prove (by Liouville's Thm) that g is constant
$\Rightarrow p$ is constant $\times$.

[If $|z|$ is large, $|g(z)|$ is small ← can bound $|g(z)|$ for $|z| \geq R$.]

$\lim_{|z| \to \infty} |g(z)| = 0$

Extreme Value theorem gives a bound for $\underbrace{|g(z)|}_{\text{continuous}}$ for $\underbrace{|z| \leq R}_{\text{cpt set}}$.

Cauchy Inequalities

CIFD
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(w)}{(w - z)^{n+1}} \, dw$$

f is entire,
z fixed $\in \mathbb{C}$.

$C_R = \{w : |z - w| = R\}$, ccw

$$\Rightarrow \left|f^{(n)}(z)\right| = \frac{n!}{2\pi}\left|\int_{C_R}\frac{f(w)\,dw}{(w-z)^{n+1}}\right|$$

ML ≤

$$\leq \frac{n!}{2\pi}\max_{w\in C_R}\left|\frac{f(w)}{(w-z)^{n+1}}\right|\cdot 2\pi R$$

$$= \frac{n!}{\cancel{2\pi}}\,\frac{\max_{w\in C_R}|f(w)|\;\cancel{2\pi}\cancel{R}}{R^{\cancel{n+1}}}$$

Cauchy Inequality

Let f be holom on $\mathbb{C}$, $z\in\mathbb{C}$,

$M_R = \max_{|w-z|=R}|f(w)|$. Then

$$\left|f^{(n)}(z)\right| \leq \frac{n!\,M_R}{R^n} \quad \text{for all } R>0.$$

Another consequence of CIFD:

☆ Every holomorphic fcn is C^∞ on its domain.

$$f \text{ holom} \Rightarrow f^{(n)}(z) = \frac{n!}{2\pi i}\int_\alpha \frac{f(w)\,dw}{(w-z)^{n+1}}$$

$\therefore$ the n^{th} derivative exists $\Rightarrow$ all derivatives exist, $(n-1)^{st}$ deriv is continuous $\Rightarrow$ all the derivatives are continuous $\Rightarrow$ f is C^∞.